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On two matrix derivatives by Kollo and von Rosen

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Abstract

The article establishes relationships between the matrix derivatives of *F* with respect to *X* as introduced by von Rosen (1988), Kollo and von Rosen (2000) and the Magnus-Neudecker (1999) matrix derivative. The usual transformations apply and the Moore-Penrose inverse of the duplication matrix is used. Both *X* and *F* have the same dimension.

MSC: primary 15A69, secondary 62H10

Keywords: Matrix derivatives and differentials, vectorization, commutation and duplication matrices

1 Introduction

Von Rosen (1988) and Kollo and von Rosen (2000) study moments of the inverted Wishart distribution. For finding specific expressions they use two types of matrix derivatives. Unfortunately these are not easily accessible to the uninitiated reader. The two are obviously related, both being matrix representations of the Fréchet derivative. There is, however, a more accessible representation, namely the Magnus-Neudecker matrix derivative. In this article we shall link the three representations and consider some illustrative applications, most of these taken from the two quoted articles.

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2 The three matrix derivatives

Von Rosen (1988) defined the matrix derivative

$$\frac{\partial F}{\partial X} = \sum_{ijkl} \varepsilon_{kl} \frac{\partial f_{ij}}{\partial x_{kl}} E_{ij} \otimes E_{kl} \tag{1}$$

where

$$\varepsilon_{kl} = \begin{cases} 1 & \text{if } k = l \\ \frac{1}{2} & \text{if } k \neq l, \end{cases} \qquad E_{ij} = e_i e_j'.$$

where e_i is the i^{th} column of the identity matrix I_p, $F = \left(f_{ij}\right)$ and $X = (x_{kl})$ are *symmetric* matrices of dimension p, and $i, j, k, l = 1, \ldots, p$. $F = F(X)$ is a function of X.

Kollo and von Rosen (2000) defined the matrix derivative

$$\frac{dF}{dX} = \sum_{ijkl} \varepsilon_{kl} \frac{\partial f_{ij}}{\partial x_{kl}} \left(\operatorname{vec} E_{ij}\right)(\operatorname{vec} E_{kl})' . \tag{2}$$

It is clear that

$$K_{p^2,p^2} \operatorname{vec} \frac{dF}{dX} = C_2^p \operatorname{vec} \frac{\partial F}{\partial X} \tag{3}$$

with $C_2^p = I_p \otimes K_{pp} \otimes I_p$, K denoting a commutation matrix. We used property (*ii*) of the Appendix.

Therefore

$$\operatorname{vec} \frac{\partial F}{\partial X} = \left(K_{pp} \otimes K_{pp}\right) C_2^p \operatorname{vec} \frac{dF}{dX}, \tag{4}$$

because

$$K_{p^2,p^2} = C_2^p \left(K_{pp} \otimes K_{pp}\right) C_2^p.$$

See, e.g. Ghazal and Neudecker (2000) for properties of C_2^p. Some will be reported in the Appendix.

In Section 3 we shall establish the identity

$$\frac{dF}{dX} = D_p \frac{\partial f}{\partial x'} D_p^+ \tag{5}$$

where $\frac{\partial f}{\partial x'}$ is the Magnus-Neudecker matrix derivative and D_p^+ is the Moore-Penrose inverse of the duplication matrix D_p. Further $f = D_p^+ \operatorname{vec} F$ and $x = D_p^+ \operatorname{vec} X$. Equivalently $f = v(F)$ and $x = v(X)$.

The combination of (4) and (5) would enable us to express $\frac{\partial F}{\partial X}$ in $\frac{\partial f}{\partial x'}$, in vectorized form. As this is not so fruitful, we shall not do it. We prefer to use (4) and subsequently devectorize the resulting $\operatorname{vec} \frac{\partial F}{\partial X}$.

3 The link between $\frac{dF}{dX}$ and $\frac{\partial f}{\partial x'}$

The first thing to do is to establish

Lemma 1

$$K_{pp}\frac{dF}{dX} = \frac{dF}{dX}, \quad \frac{dF}{dX}K_{pp} = \frac{dF}{dX}. \tag{6}$$

Proof. We have $K_{pp}(e_j \otimes e_i) = e_i \otimes e_j$. Hence

$$K_{pp}\frac{\partial f_{ij}}{\partial x_{kl}}\left(e_j \otimes e_i\right) = \frac{\partial f_{ij}}{\partial x_{kl}}\left(e_i \otimes e_j\right) = \frac{\partial f_{ji}}{\partial x_{kl}}\left(e_i \otimes e_j\right),$$

and

$$K_{pp}\sum_{ij}\frac{\partial f_{ij}}{\partial x_{kl}}\left(e_j \otimes e_i\right) = \sum_{ij}\frac{\partial f_{ji}}{\partial x_{kl}}\left(e_i \otimes e_j\right) = \sum_{ij}\frac{\partial f_{ij}}{\partial x_{kl}}\left(e_j \otimes e_i\right)$$

by interchanging the indices i and j. Hence $K_{pp}\frac{dF}{dX} = \frac{dF}{dX}$. Similarly we find that

$$\frac{dF}{dX}K_{pp} = \frac{dF}{dX}. \qquad \square$$

Having found these basic properties of $\frac{dF}{dX}$ we shall prove

Lemma 2

$$\frac{dF}{dX} d\,vec\,X = d\,vec\,F.$$

Proof. Using the definition of $\frac{dF}{dX}$ we get

$$\begin{aligned}
\frac{dF}{dX} d \operatorname{vec} X &= \sum_{ijkl} \varepsilon_{kl} \frac{\partial f_{ij}}{\partial x_{kl}} (e_j \otimes e_i)(e_l \otimes e_k)' d \operatorname{vec} X \\
&= \sum_{ijkl} \varepsilon_{kl} \frac{\partial f_{ij}}{\partial x_{kl}} \left(e_j e_l' \otimes e_i e_k' \right) d \operatorname{vec} X \\
&= \sum_{ijkl} \varepsilon_{kl} \frac{\partial f_{ij}}{\partial x_{kl}} d \operatorname{vec} e_i e_k' X e_l e_j' \\
&= \sum_{ijkl} \varepsilon_{kl} \left(\frac{\partial f_{ij}}{\partial x_{kl}} d x_{kl} \right) \operatorname{vec} e_i e_j' \\
&= \sum_{ijkl} \varepsilon_{kl} \frac{\partial \operatorname{vec} f_{ij} e_i e_j'}{\partial x_{kl}} d x_{kl} \\
&= \sum_{kl} \varepsilon_{kl} \frac{\partial \operatorname{vec} F}{\partial x_{kl}} d x_{kl} = d \operatorname{vec} F.
\end{aligned}$$

□

Having established this result we shall prove

Theorem 3

$$D_p \frac{\partial f}{\partial x'} = \frac{dF}{dX} D_p.$$

Proof. We rewrite the result of lemma 2 as

$$\frac{dF}{dX} D_p \, dx = D_p \, df = D_p \frac{\partial f}{\partial x'} dx.$$

Omitting the arbitrary dx we obtain the result. □

The basic result follows as a corollary, namely.

Corollary 4

$$\frac{dF}{dX} = D_p \frac{\partial f}{\partial x'} D_p^+.$$

Proof. Postmultiplication of the result of theorem 3 by D_p^+ yields, by virtue of lemma 1,

$$D_p \frac{\partial f}{\partial x'} D_p^+ = \frac{dF}{dX} D_p D_p^+ = \frac{1}{2} \frac{dF}{dX} \left(I_{p^2} + K_{pp} \right) = \frac{dF}{dX}.$$

□

4 Some applications of $\dfrac{dF}{dX}$

We shall examine three cases.

(1) Kollo & von Rosen (2000) find

$$\frac{dF}{dX} = \frac{1}{2}\left(I_{p^2} + K_{pp}\right),$$

for $F = X$ (their result 2.9).

This can be derived succinctly in the following way. Differentiation of F yields $dF = dX$, which leads to $d\,\mathrm{vec}\,F = d\,\mathrm{vec}\,X$ and subsequently to $df = dx$. Hence $\dfrac{dF}{dX} = D_p D_p^+ = \frac{1}{2}\left(I_{p^2} + K_{pp}\right)$ by corollary 4.

□

(2) $$\frac{dF}{dX} = -\tfrac{1}{2}\left(I_{p^2} + K_{pp}\right)\left(X^{-1} \otimes X^{-1}\right) \text{ for } F = X^{-1}.$$

Proceeding as before we get

$$\begin{aligned}
dF &= -X^{-1}(dX)X^{-1}, \\
d\,\mathrm{vec}\,F &= -\left(X^{-1} \otimes X^{-1}\right) d\,\mathrm{vec}\,X, \\
df &= -D_p^+\left(X^{-1} \otimes X^{-1}\right) D_p\, dx \quad \text{and} \\
\frac{\partial f}{\partial x'} &= -D_p^+\left(X^{-1} \otimes X^{-1}\right) D_p.
\end{aligned}$$

Corollary 4 yields then

$$\begin{aligned}
\frac{dF}{dX} &= -D_p D_p^+\left(X^{-1} \otimes X^{-1}\right) D_p D_p^+ = -\tfrac{1}{4}\left(I_{p^2} + K_{pp}\right)\left(X^{-1} \otimes X^{-1}\right)\left(I_{p^2} + K_{pp}\right) = \\
&= -\tfrac{1}{4}\left(I_{p^2} + K_{pp}\right)^2\left(X^{-1} \otimes X^{-1}\right) = -\tfrac{1}{2}\left(I_{p^2} + K_{pp}\right)\left(X^{-1} \otimes X^{-1}\right).
\end{aligned}$$

□

(3) $$\frac{dF}{dX} = \tfrac{1}{2}\left(I_{p^2} + K_{pp}\right)\left(I_p \otimes X + X \otimes I_p\right) \text{ for } F = X^2.$$

We get

$$\begin{aligned}
dF &= (dX)X + X\,dX, \\
d\,\mathrm{vec}\,F &= \left(I_p \otimes X + X \otimes I_p\right) d\,\mathrm{vec}\,X, \\
df &= D_p^+\left(I_p \otimes X + X \otimes I_p\right) D_p\, d\,x.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial f}{\partial x'} &= D_p^+\left(I_p \otimes X + X \otimes I_p\right) D_p, \\
\frac{dF}{dX} &= D_p\, D_p^+\left(I_p \otimes X + X \otimes I_p\right) D_p\, D_p^+ = \\
&= \tfrac{1}{4}\left(I_{p^2} + K_{pp}\right)\left(I_p \otimes X + X \otimes I_p\right)\left(I_{p^2} + K_{pp}\right) = \\
&= \tfrac{1}{4}\left(I_{p^2} + K_{pp}\right)^2\left(I_p \otimes X + X \otimes I_p\right) = \\
&= \tfrac{1}{2}\left(I_{p^2} + K_{pp}\right)\left(I_p \otimes X + X \otimes I_p\right).
\end{aligned}$$

□

In the following section we shall give some applications of $\dfrac{\partial F}{\partial X}$.

5 Some applications of $\dfrac{\partial F}{\partial X}$

(1) $\dfrac{\partial X}{\partial X} = \tfrac{1}{2}\left(\operatorname{vec} I_p\right)\left(\operatorname{vec} I_p\right)' + \tfrac{1}{2} K_{pp}$.

See von Rosen (1988, Lemma 2.1, *d*, *i*).

Derivation: Use $\dfrac{dX}{dX} = \tfrac{1}{2}\left(I_{p^2} + K_{pp}\right)$.

See section 4 (1). It is easy to see that $C_2^p \operatorname{vec} I_{p^2} = C_2^p \operatorname{vec}\left(I_p \otimes I_p\right) = \operatorname{vec} I_p \otimes \operatorname{vec} I_p$. Further $C_2^p \operatorname{vec} K_{pp} = \operatorname{vec} K_{pp}$. For these properties we refer to (*ii*) and (*iii*) in the Appendix.

Hence

$$\begin{aligned}
\operatorname{vec}\frac{\partial X}{\partial X} &= \tfrac{1}{2}\left(K_{pp} \otimes K_{pp}\right)\left(\operatorname{vec} I_p \otimes \operatorname{vec} I_p + \operatorname{vec} K_{pp}\right) \\
&= \tfrac{1}{2}\left(\operatorname{vec} I_p \otimes \operatorname{vec} I_p + \operatorname{vec} K_{pp}\right) \\
&= \tfrac{1}{2}\operatorname{vec}\left[\left(\operatorname{vec} I_p\right)\left(\operatorname{vec} I_p\right)' + K_{pp}\right],
\end{aligned}$$

from which (1) follows.

□

(2) $\dfrac{\partial X^{-1}}{\partial X} = -\tfrac{1}{2}\left(\operatorname{vec} X^{-1}\right)\left(\operatorname{vec} X^{-1}\right)' - \tfrac{1}{2} K_{pp}\left(X^{-1} \otimes X^{-1}\right)$.

See von Rosen (1988, Lemma 2.2*i*).

Derivation: Use $\dfrac{dX^{-1}}{dX} = -\tfrac{1}{2}\left(I_{p^2} + K_{pp}\right)\left(X^{-1} \otimes X^{-1}\right)$.

See section 4 (2). Then

$$\begin{aligned}
\operatorname{vec}\frac{\partial X^{-1}}{\partial X} &= -\tfrac{1}{2}\left(K_{pp} \otimes K_{pp}\right) C_2^p \operatorname{vec}\left[\left(I_{p^2} + K_{pp}\right)\left(X^{-1} \otimes X^{-1}\right)\right] \\
&= -\tfrac{1}{2}\left(K_{pp} \otimes K_{pp}\right)\left(\operatorname{vec} X^{-1} \otimes \operatorname{vec} X^{-1}\right) \\
&\quad -\tfrac{1}{2}\left(K_{pp} \otimes K_{pp}\right) \operatorname{vec} K_{pp}\left(X^{-1} \otimes X^{-1}\right) \\
&= -\tfrac{1}{2} \operatorname{vec} X^{-1} \otimes \operatorname{vec} X^{-1} - \tfrac{1}{2} \operatorname{vec} K_{pp}\left(X^{-1} \otimes X^{-1}\right) \\
&= -\tfrac{1}{2} \operatorname{vec}\left[\left(\operatorname{vec} X^{-1}\right)\left(\operatorname{vec} X^{-1}\right)'\right] - \tfrac{1}{2} \operatorname{vec} K_{pp}\left(X^{-1} \otimes X^{-1}\right),
\end{aligned}$$

from which (2) follows.

We also used property 18 in Neudecker (2000). For a simple proof see (*iv*) in the Appendix.

□

(3) $\dfrac{\partial X^2}{\partial X} = \tfrac{1}{2} K_{pp}\left(I_p \otimes X + X \otimes I_p\right) + \tfrac{1}{2}(\operatorname{vec} X)\left(\operatorname{vec} I_p\right)' + \tfrac{1}{2}\left(\operatorname{vec} I_p\right)(\operatorname{vec} X)'$.

Derivation: Use $\dfrac{dX^2}{dX} = \tfrac{1}{2}\left(I_{p^2} + K_{pp}\right)\left(I_p \otimes X + X \otimes I_p\right)$.

See section 4 (3). Then

$$\begin{aligned}
\operatorname{vec}\frac{\partial X^2}{\partial X} &= \tfrac{1}{2}\left(K_{pp} \otimes K_{pp}\right) C_2^p \operatorname{vec}\left[\left(I_{p^2} + K_{pp}\right)\left(I_p \otimes X + X \otimes I_p\right)\right] \\
&= \tfrac{1}{2}\left(K_{pp} \otimes K_{pp}\right)\left(\operatorname{vec} I_p \otimes \operatorname{vec} X + \operatorname{vec} X \otimes \operatorname{vec} I_p\right) \\
&+ \tfrac{1}{2}\left(K_{pp} \otimes K_{pp}\right) \operatorname{vec}\left[K_{pp}\left(I_p \otimes X + X \otimes I_p\right)\right] \\
&= \tfrac{1}{2}\left(\operatorname{vec} I_p \otimes \operatorname{vec} X + \operatorname{vec} X \otimes \operatorname{vec} I_p\right) + \tfrac{1}{2} \operatorname{vec}\left[K_{pp}\left(I_p \otimes X + X \otimes I_p\right)\right] \\
&= \tfrac{1}{2} \operatorname{vec}\left[(\operatorname{vec} X)\left(\operatorname{vec} I_p\right)' + \left(\operatorname{vec} I_p\right)(\operatorname{vec} X)'\right] \\
&+ \tfrac{1}{2} \operatorname{vec}\left[K_{pp}\left(I_p \otimes X + X \otimes I_p\right)\right].
\end{aligned}$$

Devectorization yields (3).

□

(4) $\dfrac{\partial F^{-1}}{\partial X} = -\left(F^{-1} \otimes I_p\right) \dfrac{\partial F}{\partial X}\left(F^{-1} \otimes I_p\right)$, where $F = F(X)$.

See von Rosen (1988, Lemma 2.1, *c*, *iii*).

Derivation: It is known that $dF^{-1} = -F^{-1}(dF)F^{-1}$, hence

$$\begin{aligned} dv\left(F^{-1}\right) &= D_p^+ \, d \operatorname{vec} F^{-1} = -D_p^+ \left(F^{-1} \otimes F^{-1}\right) d \operatorname{vec} F \\ &= -D_p^+ \left(F^{-1} \otimes F^{-1}\right) D_p \, d\, v(F) = -D_p^+ \left(F^{-1} \otimes F^{-1}\right) D_p \frac{\partial f}{\partial x'} dx \end{aligned}$$

and finally

$$\frac{\partial v\left(F^{-1}\right)}{\partial x'} = -D_p^+ \left(F^{-1} \otimes F^{-1}\right) D_p \frac{\partial f}{\partial x'}.$$

By Corollary 4 and Lemma 1 we have

$$\begin{aligned} \frac{dF^{-1}}{dX} &= -D_p \, D_p^+ \left(F^{-1} \otimes F^{-1}\right) D_p \frac{\partial f}{\partial x'} D_p^+ \\ &= -\tfrac{1}{2} \left(I_{p^2} + K_{pp}\right) \left(F^{-1} \otimes F^{-1}\right) \frac{dF}{dX} \\ &= -\tfrac{1}{2} \left(F^{-1} \otimes F^{-1}\right) \left(I_{p^2} + K_{pp}\right) \frac{dF}{dX} \\ &= -\left(F^{-1} \otimes F^{-1}\right) \frac{dF}{dX}. \end{aligned}$$

Application of Section 2 (4) yields

$$\begin{aligned} \operatorname{vec} \frac{\partial F^{-1}}{\partial X} &= -\left(K_{pp} \otimes K_{pp}\right) C_2^p \operatorname{vec} \left(F^{-1} \otimes F^{-1}\right) \frac{dF}{dX} \\ &= -\left(K_{pp} \otimes K_{pp}\right) C_2^p \left(I_{p^2} \otimes F^{-1} \otimes F^{-1}\right) \operatorname{vec} \frac{dF}{dX} \\ &= -\left(K_{pp} \otimes K_{pp}\right) C_2^p \left(I_p \otimes I_p \otimes F^{-1} \otimes F^{-1}\right) \operatorname{vec} \frac{dF}{dX} \\ &= -\left(K_{pp} \otimes K_{pp}\right) \left(I_p \otimes F^{-1} \otimes I_p \otimes F^{-1}\right) C_2^p \operatorname{vec} \frac{dF}{dX} \\ &= -\left(F^{-1} \otimes I_p \otimes F^{-1} \otimes I_p\right) \left(K_{pp} \otimes K_{pp}\right) C_2^p \operatorname{vec} \frac{dF}{dX} \\ &= -\left(F^{-1} \otimes I_p \otimes F^{-1} \otimes I_p\right) \operatorname{vec} \frac{\partial F}{\partial X} \\ &= -\operatorname{vec} \left[\left(F^{-1} \otimes I_p\right) \frac{\partial F}{\partial X} \left(F^{-1} \otimes I_p\right) \right]. \end{aligned}$$

Hence

$$\frac{\partial F^{-1}}{\partial X} = -\left(F^{-1} \otimes I_p\right) \frac{\partial F}{\partial X} \left(F^{-1} \otimes I_p\right).$$

□

6 Appendix

The following matrix properties have been used in this article. The first five involve $C_2^p = I_p \otimes K_{pp} \otimes I_p$.

(i) $C_2^p (A \otimes B \otimes C \otimes D)\, C_2^p = A \otimes C \otimes B \otimes D$, for $(p \times p)\, A, B, C, D$.

(ii) $C_2^p \operatorname{vec}(A \otimes B) = \operatorname{vec} A \otimes \operatorname{vec} B$, for $(p \times p)\, A$ and B.

Proof.

$$\begin{aligned}
C_2^p \operatorname{vec}(A \otimes B) &= \sum_{ij} \left(I_p \otimes E_{ij} \otimes E_{ji} \otimes I_p\right) \operatorname{vec}(A \otimes B) \\
&= \sum_{ij} \operatorname{vec}\left[\left(E_{ji} \otimes I_p\right)(A \otimes B)\left(I_p \otimes E_{ji}\right)\right] \\
&= \sum_{ij} \operatorname{vec}\left(E_{ji} A \otimes B E_{ji}\right) = \sum_{ij} \operatorname{vec}\left(e_j A_{i.} \otimes B_{.j} e_i'\right) \\
&= \sum_{ij} \operatorname{vec}\left[\left(e_j \otimes B_{.j}\right)(A_{i.} \otimes e_i')\right] = \\
&= \operatorname{vec}\left[\left(\sum_j \operatorname{vec} B_{.j} e_j'\right)\left(\sum_i \operatorname{vec} e_i A_{i.}\right)'\right] \\
&= \operatorname{vec}\left[(\operatorname{vec} B)(\operatorname{vec} A)'\right] = \operatorname{vec} A \otimes \operatorname{vec} B.
\end{aligned}$$

As usual $A_{i.}$ is the i^{th} row of A, $A_{.j}$ is the j^{th} column of A.

□

(iii) $C_2^p \operatorname{vec} K_{pp} = \operatorname{vec} K_{pp}$.

Proof.

$$\begin{aligned}
C_2^p \operatorname{vec} K_{pp} &= \sum_{ij} C_2^p \operatorname{vec}\left(E_{ij} \otimes E_{ji}\right) \\
&= \sum_{ij} \left(\operatorname{vec} E_{ij} \otimes \operatorname{vec} E_{ji}\right) = \operatorname{vec} \sum_{ij} \left(\operatorname{vec} E_{ji}\right)\left(\operatorname{vec} E_{ij}\right)' \\
&= \operatorname{vec} \sum_{ij} \left(e_i \otimes e_j\right)\left(e_j' \otimes e_i'\right) = \operatorname{vec} \sum_{ij} \left(e_i e_j' \otimes e_j e_i'\right) \\
&= \operatorname{vec} \sum_{ij} \left(E_{ij} \otimes E_{ji}\right) = \operatorname{vec} K_{pp}.
\end{aligned}$$

□

(iv) $C_2^p \operatorname{vec} K_{pp} (A \otimes B) = \operatorname{vec} K_{pp} (A \otimes B')$ for $(p \times p)\, A$ and B.

Proof.

$$
\begin{aligned}
C_2^p \operatorname{vec} K_{pp} (A \otimes B) &= C_2^p \left(A' \otimes B' \otimes I_p \otimes I_p\right) \operatorname{vec} K_{pp} \\
&= \left(A' \otimes I_p \otimes B' \otimes I_p\right) C_2^p \operatorname{vec} K_{pp} = \left(A' \otimes I_p \otimes B' \otimes I_p\right) \operatorname{vec} K_{pp} \\
&= \operatorname{vec} \left(B' \otimes I_p\right) K_{pp} \left(A \otimes I_p\right) = \operatorname{vec} K_{pp} \left(I_p \otimes B'\right) \left(A \otimes I_p\right) \\
&= \operatorname{vec} K_{pp} (A \otimes B') .
\end{aligned}
$$

□

(v) $K_{p^2,p^2} = C_2^p \left(K_{pp} \otimes K_{pp}\right) C_2^p.$

Proof. Consider the $\left(p^2,\ p^2\right)$ matrix

$$Q = \sum_{kl} (E_{kl} \otimes Q_{kl})$$

or equivalently

$$Q = [Q_{kl}] \quad (k, l = 1, \ldots, p)$$

with $(p \times p)$ matrix Q_{kl}.

Then $K_{p^2,p^2} \operatorname{vec} Q = \operatorname{vec} Q'$ and

$$
\begin{aligned}
C_2^p \left(K_{pp} \otimes K_{pp}\right) C_2^p \operatorname{vec} Q &= \sum_{ijst} C_2^p \left(E_{ij} \otimes E_{ji} \otimes E_{st} \otimes E_{ts}\right) C_2^p \operatorname{vec} Q \\
&= \sum_{ijstkl} \left(E_{ij} \otimes E_{st} \otimes E_{ji} \otimes E_{ts}\right) \operatorname{vec} (E_{kl} \otimes Q_{kl}) \\
&= \operatorname{vec} \sum_{ijstkl} \left(E_{ji} \otimes E_{ts}\right) (E_{kl} \otimes Q_{kl}) \left(E_{ji} \otimes E_{ts}\right) \\
&= \operatorname{vec} \sum_{ijstkl} \left(E_{ji}\, E_{kl}\, E_{ji} \otimes E_{ts}\, Q_{kl}\, E_{ts}\right) \\
&= \operatorname{vec} \sum_{ijst} \left(E_{ji} \otimes E_{ts}\, Q_{ij}\, E_{ts}\right) = \operatorname{vec} \sum_{ijst} \left(Q_{ij}\right)_{st} \left(E_{ij} \otimes E_{st}\right)' \\
&= \operatorname{vec} \sum_{ij} \left(E_{ij} \otimes Q_{ij}\right)' = \operatorname{vec} Q',
\end{aligned}
$$

where $\left(Q_{ij}\right)_{st}$ is the (s, t) element of Q_{ij}.

□

We further used the standard properties:

(vi) $K_{pp} \operatorname{vec} A = \operatorname{vec} A'$ for $(p \times p)$ matrix A.

(vii) $D_p D_p^+ = \frac{1}{2}\left(I_{p^2} + K_{pp}\right),\ D_p^+ D_p = I_{p^*}$ with $2p^* = p(p+1)$.

$(viii)$ $\text{vec}\, A\, B\, C = (C' \otimes A)\ \text{vec}\, B$ for compatible matrices A, B and C.

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Resum

Aquest article estableix les relacions entre las derivades matricials de *F* respecte de *X* introduïdes per von Rosen (1988), Kollo i von Rosen (2000) i les derivades matricials de Magnus i Neudecker (1999). Les operacions vectorials de duplicació i transformació en vectors són les usuals i les inverses de las matrius duplicades són les de Moore-Penrose. Ambdues *X* i *F = F(X)* tenen la mateixa dimensió.

MSC: primary 15A69, secondary 62H10

Paraules clau: Derivades i diferencials de matrius, vectorització, commutació i duplicació de matrius